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Chebyshev Approximation of Plane Curves by Splines

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Given a parametric plane curve **p** and any Bézier curve **q** of degree n such that **p** and **q** have contact of order k at the common end points, we use the normal vector field of **p** to measure the distance of corresponding points of **p** and **q**. Applying the theory of nonlinear Chebyshev approximation, we show that the maximum norm of this distance (or error) function $\rho_{\mathbf{q}}$ is locally minimal for **q** if and only if $\rho_{\mathbf{q}}$ is an alternant with $2 \cdot (n-k-1)+1$ extreme points. Finally, a Remes type algorithm is presented for the numerical computation of a locally best approximation to **p**.

INTRODUCTION

In CAGD, several methods have been developed for the approximation of a given regular parametric curve $\mathbf{p}: [0, 1] \to \mathbb{R}^2$ by a Bézier curve **q** of fixed degree n, where **p** and **q** are supposed to have contact of order k at the end points (see, e.g., Hölzle [8], Hoschek [9], de Boor *et al.* [1], and for circular arcs, Dokken *et al.* [5] and Goldapp [7]). Assuming that there is a unique continuous reparametrization $\varphi: [0, 1] \to [0, 1]$ such that for each $s \in [0, 1]$ the point $\mathbf{q}(\varphi(s))$ lies on the oriented normal of **p** at s, W. Degen [4] has introduced the error function $\rho_{\mathbf{q}}: s \mapsto \pm |\mathbf{q}(\varphi(s)) - \mathbf{p}(s)|_2$. In many cases the maximum norm $|\rho_{\mathbf{q}}|_\infty$ of $\rho_{\mathbf{q}}$ is equal to the Hausdorff distance of **p** and **q** (see Degen [4]), and we call **q** a best approximation to **p** if $|\rho_{\mathbf{q}}|_\infty$ is minimal for the Bézier curve **q**. In some cases Degen has characterized a best approximation **q** to **p** by an alternation property of the error function $\rho_{\mathbf{q}}$ by using a theorem of Meinardus and Schwedt.

In this paper we prove for arbitrary $n \in \mathbb{N}$ and all possible $k \in \mathbb{N}_0$ that a Bézier curve **q** is a locally best approximation to **p** if and only if the error function $\rho_{\mathbf{q}}$ alternates at $2 \cdot (n-k-1)+1$ extreme points and that a locally best approximation is locally unique (Theorem 2.9). In particular, we show that two approximating Bézier curves locally have only $2n-1$ intersections

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(Theorem 2.7). Moreover, a best approximation can be calculated by a Remes type algorithm (see Section 2.11).

In Section 1 we introduce the class of admissible curves for which the error function and the normal distance to $\mathbf{p}$ are defined. In Section 2 we modify the nonlinear theory of Rice [15] and then apply it to our family of error functions. This leads to a nonlinear system of equations, discussed in Section 3. Finally, our main results are proved in Section 4.

1. Notations, the Error Function, and the Normal Distance

Henceforth, let I denote the unit interval $[0, 1]$, and $C[0, 1]$ is understood to be the space of continuous functions $f: I \to \mathbb{R}$ endowed with the uniform norm $|f|_\infty := \max_{t \in I} |f(t)|$. If k is a positive integer, then any at least k-times continuously differentiable map $\mathbf{x}: I \to \mathbb{R}^2$ is called C^k *curve*, a continuous map $\mathbf{x}: I \to \mathbb{R}^2$ is called C^0 *curve*, and an at least k-times continuously differentiable bijective function $\alpha: I \to I$ such that $\alpha'(t) > 0$ for every $t \in I$ is called C^k *reparametrization*. Finally, a C^1 curve $\mathbf{x}$ satisfying $\dot{\mathbf{x}}(s) := (d\mathbf{x}/dt)(s) \neq (0, 0)$ for every $s \in I$ is called *regular curve*. The following parametrization-independent concept of contact of two curves at a common end point, also known as geometric continuity, is crucial in CAGD when fitting together several curve segments to a whole one (cf. [10, 5]).

Definition (cf. [3, 3.3.2]). Let $s \in \{0, 1\}$ and $k \in \mathbb{N}_0$. Two C^k curves $\mathbf{x}$ and $\mathbf{y}$ have *contact of order k at s* iff $\mathbf{x}(s) = \mathbf{y}(s)$ and in case $k > 0$ there are C^k reparametrizations α and β such that

$$\frac{d^j(\mathbf{x} \circ \alpha)}{dt^j}(s) = \frac{d^j(\mathbf{y} \circ \beta)}{dt^j}(s), \qquad j = 1, ..., k.$$

For example, two C^1 curves $\mathbf{x}$ and $\mathbf{y}$ have contact of order one at 0 and 1 if and only if $\mathbf{x}$ and $\mathbf{y}$ have common end points and there are λ, $\mu > 0$ such that $\dot{\mathbf{x}}(0) = \lambda \cdot \dot{\mathbf{y}}(0)$ and $\dot{\mathbf{x}}(1) = \mu \cdot \dot{\mathbf{y}}(1)$.

For the remainder of this paper, let $\mathbf{p} = (p_1, p_2): I := [0, 1] \to \mathbb{R}^2$ be a fixed regular curve (injectivity is not supposed) to be approximated. Using the normal vector field of $\mathbf{p}$, we now introduce the class of admissible curves with respect to $\mathbf{p}$ for which the normal distance to $\mathbf{p}$ can be defined:

1.2. Definition. A regular curve $\mathbf{q}: I \to \mathbb{R}^2$ is said to be *admissible* with respect to $\mathbf{p}$ if and only if

(1) $\mathbf{q}(s) = \mathbf{p}(s)$, $s = 0, 1$.

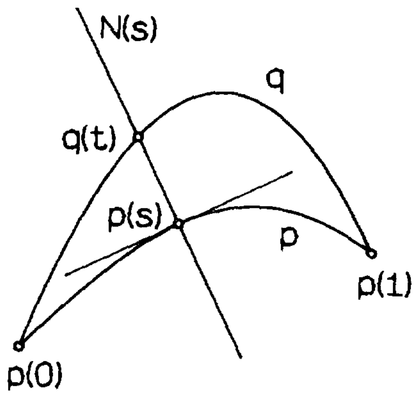

FIG. 1. Admissible parabola q.

(2) There exists a unique strictly increasing bijective map $\varphi_{\mathbf{q}}: I \to I$ such that, for each $s \in I$, the point $\mathbf{q}(\varphi_{\mathbf{q}}(s))$ lies on the normal $N(s) := \{\mathbf{p}(s) + t \cdot \mathbf{n}(s) \mid t \in \mathbb{R}\}$ of $\mathbf{p}$ at s, where $\mathbf{n}(s)$ denotes the unit normal vector $|\dot{\mathbf{p}}|_2^{-1} \cdot (-\dot{p}_2, \dot{p}_1)$ of $\mathbf{p}$ at s.

(3) The tangent vector $\dot{\mathbf{q}}(\varphi_{\mathbf{q}}(s))$ of $\mathbf{q}$ is not parallel to $\mathbf{n}(s)$ for every $s \in I$.

If $\mathbf{q}$ is an admissible curve (Fig. 1), then there is a unique continuous map $\rho_{\mathbf{q}}: I \to \mathbb{R}$ satisfying $\mathbf{q}(\varphi_{\mathbf{q}}(s)) = \mathbf{p}(s) + \rho_{\mathbf{q}}(s) \cdot \mathbf{n}(s)$ for every $s \in I$, where $\rho_{\mathbf{q}}(0) = \rho_{\mathbf{q}}(1) = 0$. This implies $|\rho_{\mathbf{q}}(s)| = |\mathbf{q}(\varphi_{\mathbf{q}}(s)) - \mathbf{p}(s)|_2$, $s \in I$, where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. Hence we define (see Degen [4, Definition 2]):

1.3. DEFINITION. Let $\mathbf{q}$ be an admissible curve. Then the map $\rho_{\mathbf{q}}$ is called *error function of* $\mathbf{q}$ with respect to $\mathbf{p}$ and $d(\mathbf{q}) := |\rho_{\mathbf{q}}|_\infty$ *normal distance from* $\mathbf{q}$ *to* $\mathbf{p}$.

1.4. *Remarks.* (a) Using the implicit function theorem, one can easily show that, for each admissible curve $\mathbf{q}$, the functions $\varphi_{\mathbf{q}}$ and $\rho_{\mathbf{q}}$ are at least of class C^1 if $\mathbf{p}$ is at least of class C^2.

(b) If $\mathbf{p}$ is a segment of the unit circle (cf. [5, p. 35; 7]), then $d(\mathbf{q}) = |\,|\mathbf{q}|_2 - 1|_\infty$ for each admissible curve $\mathbf{q}$.

(c) $\rho_{\mathbf{q}}$ is constant if and only if $\rho_{\mathbf{q}} \equiv 0$.

We now introduce some classes of admissible polynomial curves for the approximation of $\mathbf{p}$. For this, let $n \geqslant 2$ be a fixed natural number, and for any $n+1$ points $a_i \in \mathbb{R}^2$ ($i = 0, ..., n$) and $\mathbf{a} := (a_0, ..., a_n)$ let $\mathbf{P_a}$ denote the polynomial curve $\sum_{i=0}^{n} a_i \cdot t^i$ ($t \in I$). Clearly, we can write $\mathbf{P_a}$ as Bézier curve, but for our proofs it is more convenient to use Taylor expansion. For the sake of brevity, we put:

$$A := \{\mathbf{a} = (a_0, ..., a_n) \mid a_i \in \mathbb{R}^2 (i = 0, ..., n),\ a_n \neq (0, 0), \text{ and } \operatorname{res}(\dot{P}_{\mathbf{c}}, \dot{P}_{\mathbf{d}}) \neq 0 \text{ if } \mathbf{P}_{\mathbf{a}} = (P_{\mathbf{c}}, P_{\mathbf{d}})\}, \tag{1.1}$$

where $\operatorname{res}(\dot{P}_{\mathbf{c}}, \dot{P}_{\mathbf{d}})$ denotes the resultant of the two polynomial coordinate functions of the derivative $\dot{\mathbf{P}}_{\mathbf{a}}$ of $\mathbf{P}_{\mathbf{a}}$. The property $\operatorname{res}(\dot{P}_{\mathbf{c}}, \dot{P}_{\mathbf{d}}) \neq 0$ is used in the proofs of Section 4 (cf. Section 3.1) and implies that the curve $\mathbf{P}_{\mathbf{a}}$ is regular (see Walker [16, p. 24]). Moreover, it follows that A is an open subset of $\mathbb{R}^{2n+2}$. The set A and subsets of A are always endowed with the trace of the Euclidean topology.

For the remainder of this paper, let there be an open nonempty subset B of $A \subset \mathbb{R}^{2n+2}$ such that each curve $\mathbf{q} \in \{\mathbf{P}_{\mathbf{a}} \mid \mathbf{a} \in B \text{ and } \mathbf{P}_{\mathbf{a}}(s) = \mathbf{p}(s), s = 0, 1\}$ is admissible, and for $k \in \{0, ..., n-2\}$ put

$$B_k := \{\mathbf{a} \in B \mid \mathbf{P}_{\mathbf{a}} \text{ and } \mathbf{p} \text{ have contact of order } k \text{ at } 0 \text{ and } 1\}. \tag{1.2}$$

Note that the sets B_k are not open. If $k \in \{0, ..., n-2\}$ is given, we always suppose $B_k \neq \emptyset$. Furthermore, we define mappings $\phi\colon B_0 \times I \to I$ and $E\colon B_0 \times I \to \mathbb{R}$ by letting

$$\phi(\mathbf{a}, s) := \varphi_{(\mathbf{P}_{\mathbf{a}})}(s) \quad \text{and} \quad E(\mathbf{a}, s) := \rho_{(\mathbf{P}_{\mathbf{a}})}(s),\ (\mathbf{a}, s) \in B_0 \times I. \tag{1.3}$$

By [4, Theorem 2], ϕ and E are continuous. This paper addresses to the following problem: For given $k \in \{0, ..., n-2\}$ characterize each $\mathbf{a} \in B_k$ for which there is a neighborhood $U \subset B$ of $\mathbf{a}$ satisfying

$$d(\mathbf{P}_{\mathbf{a}}) = |E(\mathbf{a}, \cdot) - 0|_\infty \leqslant d(\mathbf{P}_{\mathbf{b}}) = |E(\mathbf{b}, \cdot) - 0|_\infty \qquad \text{for all} \quad \mathbf{b} \in B_k \cap U, \tag{1.4}$$

i.e., $E(\mathbf{a}, \cdot)$ is a locally best uniform approximation to $f := 0$ in B_k. This Chebyshev approximation problem is solved in the next section (see Theorem 2.9). A polynomial curve $\mathbf{P}_{\mathbf{a}}$ ($\mathbf{a} \in B_k$) satisfying (1.4) for some neighborhood U of $\mathbf{a}$ is called a *locally best GC^k-approximation to* $\mathbf{p}$. It remains open whether there actually can be several locally best GC^k-approximations.

2. A Modification of the Theory of Rice and Main Results

In this section we assume that D is any nonempty subset of $\mathbb{R}^{2n+2}$, not necessarily open, and F is any continuous mapping from $D \times I$ into $\mathbb{R}$. Given a continuous function $f\colon I \to \mathbb{R}$ and a point $a \in D$, the function

$F(a,\cdot):I\to\mathbb{R}$ is said to be a *best approximation to f with respect to F* if $|F(a,\cdot)-f|_\infty \leqslant |F(b,\cdot)-f|_\infty$ for every $b\in D$. In Section 1, we always have $E(a,0)=E(a,1)=0$ for all $a\in B$. Therefore, we assume $F(a,0)=F(a,1)=0$ for every $a\in D$, and hence we need an adaption of some definitions and theorems of Rice (see [14] or [15, pp. 3–12]). If, in addition, $f(0)=f(1)=0$, then the problem of a nonzero constant error curve $F(a,\cdot)-f$ does not arise (cf. Braess [2, 3.16]).

2.1. DEFINITION. F is called *locally solvent of degree $m\in\mathbb{N}$ at $a\in D$ with respect to* $]0,1[$ iff given m points s_j with $0<s_1<\cdots<s_m<1$ and $\varepsilon>0$ there is $\delta:=\delta(a,\varepsilon,s_1,...,s_m)>0$ such that $y_j\in\mathbb{R}$ and $|y_j-F(a,s_j)|<\delta$, $j=1,...,m$, implies the existence of $b\in D$ satisfying $F(b,s_j)=y_j$, $j=1,...,m$, and $|F(a,\cdot)-F(b,\cdot)|_\infty<\varepsilon$.

2.2. DEFINITION. F has *Property Z of degree $m\in\mathbb{N}$ at $a\in D$ with respect to* $]0,1[$ iff, for any $b\in D$, the difference $F(a,\cdot)-F(b,\cdot)$ has either at most $m-1$ zeros in $]0,1[$ or vanishes identically.

2.3. DEFINITION. F is called *varisolvent with respect to* $]0,1[$ iff, at each point $a\in D$, both the local solvency and the Property Z w.r. to $]0,1[$ are defined and have the same degree $m(a)$.

2.4. DEFINITION. A continuous function $f: I\to\mathbb{R}$ alternates (at least) $k\in\mathbb{N}$ times if and only if there are $k+1$ values $s_i\in I$ such that $0\leqslant s_1<s_2<\cdots<s_{k+1}\leqslant 1$ and $f(s_i)=-f(s_{i+1})=\pm|f|_\infty$, $i=1,...,k$.

2.5. THEOREM. *Let $f\in C[0,1]$ with $f(0)=f(1)=0$, $a\in D$, and let F be varisolvent w.r. to $]0,1[$. Then $F(a,\cdot)$ is a best approximation to f iff $F(a,\cdot)-f$ alternates at least $m(a)$ times. Furthermore, there is at most one best approximation to f.*

Proof. If $F(a,\cdot)-f$ is constant, then $F(a,s)-f(s)=F(a,0)-f(0)=0$ for every $s\in I$ and the statement is obvious. Therefore, we may assume that $F(a,\cdot)-f$ is not constant. Now the proofs of Theorems 7-3 and 7-4 in [15] (or [2, Theorems 3.9 and 3.10]) carry over. We only have to note that $|F(a,\cdot)-f|_\infty$ is not assumed at both 0 and 1; i.e., the other case can be omitted in [15, p. 11]. ∎

We now apply these definitions and theorems to restrictions of the mapping E of Section 1. The involved proofs of the following two theorems are postponed to Section 4.

2.6. THEOREM. *For each $k\in\{0,...,n-2\}$ and each open subset U of B with $V:=B_k\cap U\neq\varnothing$, the restricted mapping $E|_{(V\times I)}=:F$ is locally solvent* *of fixed degree* $2\cdot(n-k-1)$ *at each* $\mathbf{b}\in V$ *w.r.t. to* $]0,1[$. *In particular, this is valid for* $V:=B_k$.

Let $k\in\{0,...,n-2\}$ and $\mathbf{a}\in B_k$. If there is a neighborhood U of $\mathbf{a}$ in $\mathbb{R}^{2n+2}$ such that $F:=E\,|_{(V\times I)}$ with $V:=B_k\cap U$ has Property Z at each $\mathbf{b}\in V$ of fixed degree $2\cdot(n-k-1)$ or is varisolvent of fixed degree $2\cdot(n-k-1)$ w.r. to $]0,1[$, then $E\,|_{(B_k\times I)}$ is said to have *local Property Z* at $\mathbf{a}$ or to be *locally varisolvent at* $\mathbf{a}$ of fixed degree $2\cdot(n-k-1)$ w.r. to $]0,1[$. Now we have:

2.7. THEOREM. *For each* $k\in\{0,...,n-2\}$ *and* $\mathbf{a}\in B_k$, *the mapping* $F:=E\,|_{(B_k\times I)}$ *has local Property Z at* $\mathbf{a}$ *of fixed degree* $2\cdot(n-k-1)$ *w.r. to* $]0,1[$.

2.8. COROLLARY. *For each* $k\in\{0,...,n-2\}$ *and* $\mathbf{a}\in B_k$, $F:=E\,|_{(B_k\times I)}$ *is locally varisolvent at* $\mathbf{a}$ *of fixed degree* $2\cdot(n-k-1)$ *w.r. to* $]0,1[$.

Putting $f:=0$, we obtain from Theorem 2.5 a characterization and uniqueness theorem for locally best GC^k-approximations:

2.9. THEOREM. *For each* $k\in\{0,...,n-2\}$ *and* $\mathbf{a}\in B_k$, *the curve* $\mathbf{P_a}$ *is a locally best* GC^k*-approximation to the curve* $\mathbf{p}$ *if and only if* $E(\mathbf{a},\cdot)$ *alternates* $2\cdot(n-k-1)$ *times. Furthermore, any such locally best* GC^k*-approximation to* $\mathbf{p}$ *is the only one in some neighborhood of* $\mathbf{a}$.

In the cases $n=2$, $k=0$ and $n=3$, $k=1$ this theorem has been proved by Degen [4].

2.10. *Remarks.* (a) For $n=2$ it is immediate that $E\,|_{(B_0\times I)}$ has (global) Property Z of degree 2 w.r. to $]0,1[$, because each curve $\mathbf{P_a}$, $\mathbf{a}\in A$, is a parabola for $n=2$.

(b) Since $E(\mathbf{a},s)=E(\mathbf{b},s)$ is equivalent to $\mathbf{P_a}(\phi(\mathbf{a},s))=\mathbf{P_b}(\phi(\mathbf{b},s))$ for all distinct $\mathbf{a},\mathbf{b}\in B$ and $s\in I$, the number of zeros of $E(\mathbf{a},\cdot)-E(\mathbf{b},\cdot)$ in I is equal to the number of local intersection points of $\mathbf{P_a}$ and $\mathbf{P_b}$. By Bezout's theorem, this number is less than or equal to n^2, but Theorem 2.7 with $k:=0$ yields the better bound $2n-1$ in some neighborhood of $\mathbf{a}$. It remains open whether there are, in general, distinct, $\mathbf{a},\mathbf{b}\in B$ such that $E(\mathbf{a},\cdot)-E(\mathbf{b},\cdot)$ has more than $2n-1$ zeros.

(c) Let k_1, $k_2\in\{0,...,n-2\}$. Theorem 2.9 remains true if the expression $2\cdot(n-k-1)$ is replaced by $2n-k_1-k_2-2$ and B_k is replaced by the set $\{\mathbf{a}\in B\mid \mathbf{p}$ and $\mathbf{P_a}$ have contact of order k_1 at 0 and contact of order k_2 at $1\}$.

2.11. *Numerical computation.* Using compactness arguments, the existence of a (locally) best GC^k-approximation $\mathbf{P_a}$ to $\mathbf{p}$, i.e., (1.4) is satisfied, can be established in many cases (cf. [15, p. 9]), and, by Corollary 2.8, we may apply the Remes algorithm in the nonlinear case for the numerical computation of $\mathbf{P}_{\mathbf{a}}$. We describe this method in case $k := 1$ and $n \geqslant 3$. Then the solution $\mathbf{P}_{\mathbf{a}}$ admits a Bézier representation as follows: $\mathbf{P}_{\mathbf{a}}(t) = \sum_{i=0}^{n} \mathbf{b}_i \cdot \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}$, where $\mathbf{b}_0 := \mathbf{p}(0)$, $\mathbf{b}_n := \mathbf{p}(1)$, $\mathbf{b}_1 := \mathbf{b}_0 + \lambda \cdot \dot{\mathbf{p}}(0)$, $\mathbf{b}_{n-1} := \mathbf{b}_n - \mu \cdot \dot{\mathbf{p}}(1)$, and the values λ, $\mu \in \mathbb{R}$ and $\mathbf{b}_i \in \mathbb{R}^2$ $(i = 2, ..., n-2)$ are unknown. The error mapping E is not explicitly known in contrast to Meinardus [12, 8.4], but, by Theorem 2.9, there are $r := 2 \cdot (n-2) + 1$ values $s_i \in I$ with $0 < s_1 < \cdots < s_r < 1$, r values $t_i := \phi(\mathbf{a}, s_i)$, and a value $d \in \{\pm\, |E(\mathbf{a}, \cdot)|_\infty\}$ such that

$$\begin{aligned} \mathbf{p}(s_1) - d \cdot \mathbf{n}(s_1) &= \mathbf{P}_{\mathbf{a}}(t_1) \\ \mathbf{p}(s_2) + d \cdot \mathbf{n}(s_2) &= \mathbf{P}_{\mathbf{a}}(t_2) \\ &\vdots \\ \mathbf{p}(s_r) + (-1)^r \cdot d \cdot \mathbf{n}(s_r) &= \mathbf{P}_{\mathbf{a}}(t_r). \end{aligned}$$

Now this nonlinear system of equations is solved by Newton's method with $2 \cdot r$ unknowns λ, μ, d, $\mathbf{b}_i$ $(i = 2, ..., n-2)$, and t_j $(j = 1, ..., r)$ for fixed(!) values s_j $(j = 1, ..., r)$. For $n := 3$ we have used the fixed values $s_i := i/4$ and the initial values $t_i := i/4$ $(i = 1, 2, 3)$, $\lambda := \mu := \frac{1}{3}$, and $d \approx d(\mathbf{P}_{\mathbf{a}})$. If the method converges, we obtain a first approximation $\mathbf{P}_{\mathbf{a}_1}$. Then numerically calculate approximate values s_i^1 such that $E(\mathbf{a}_1, s_i^1)$ has a local minimum or maximum $(i = 1, ..., r)$. If $s_i \approx s_i^1$, the procedure stops; otherwise replace s_i by s_i^1 $(i = 1, ..., r)$ and solve the above system again. If $\mathbf{p}$ is a segment of the unit circle, this method is fast and stable, and the results are the same as in [5, p. 40].

To illustrate the error function $E(\mathbf{a}, \cdot)$ of the best GC^1-approximation $\mathbf{P}_{\mathbf{a}}$ to $\mathbf{p}$ by Bézier cubics, i.e., $n := 3$ and $k := 1$, we have calculated the best GC^1-approximation to the Trisectrix of Maclaurin (see [11, 4.7] with parametrization: $\mathbf{p}(s) = a \cdot (1 - 4 \cdot \cos^2 \bar{s}, \tan \bar{s} \cdot (1 - 4 \cdot \cos^2 \bar{s}))$, where $\bar{s} := 2.2 \cdot s - 1.1$ and $a := 1.7$. We have obtained the distance $d(\mathbf{P}_{\mathbf{a}}) \approx 1.1846$ and three extreme points of $E(\mathbf{a}, \cdot)$ (see dotted lines in Fig. 2).

3. A Nonlinear System of Equations and Its Jacobian and Divided Differences

The results of this section are needed for the proofs of Theorems 2.6 and 2.7. In particular, the proof of the local solvency leads to the system of Eqs. (3.2).

For every vector $\mathbf{a} = (a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1}$, the polynomial function $t \in \mathbb{R} \mapsto \sum_{i=0}^{n} a_i \cdot t^i$ is denoted by $P_{\mathbf{a}}$. We now introduce $4n$ polynomial

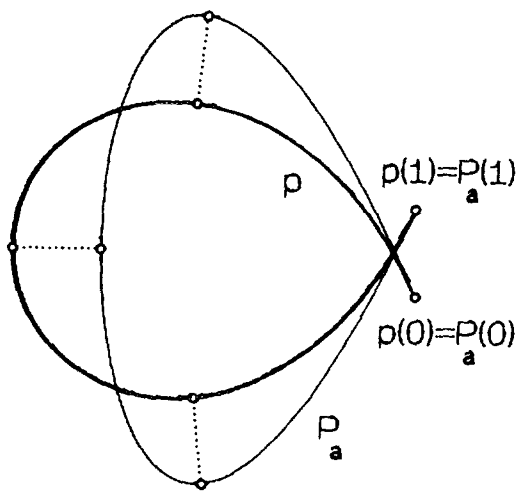

FIGURE 2

functions F_{1j}, F_{2j} from $\mathbb{R}^{4n}$ into $\mathbb{R}^{4n}$ for $j=1, ..., 2n$. The $4n$ variables are denoted by $c_0, ..., c_n, d_0, ..., d_n, t_2, ..., t_{2n-1}$, and we define $\mathbf{c} := (c_0, ..., c_n)$, $\mathbf{d} := (d_0, ..., d_n)$, $t_1 := 0$, $t_{2n} := 1$, $\mathbf{x} := (c_0, ..., c_n, d_0, ..., d_n, t_2, ..., t_{2n-1})$, and

$$\begin{gathered} F_{1j}(\mathbf{x}) := P_{\mathbf{c}}(t_j), \qquad F_{2j}(\mathbf{x}) := P_{\mathbf{d}}(t_j), \qquad j=1, ..., 2n, \\ F := (F_{1,1}, ..., F_{1,2n}, F_{2,1}, ..., F_{2,2n}). \end{gathered} \tag{3.1}$$

For given points $(u_i, v_i) \in \mathbb{R}^2$, $i=1, ..., 2n$, the following nonlinear system of equations will be of interest in Section 4 (t_1 and t_{2n} are constants!):

$$F_{1j}(\mathbf{x}) = u_j \qquad \text{and} \qquad F_{2j}(\mathbf{x}) = v_j, \qquad j=1, ..., 2n. \tag{3.2}$$

Putting $\mathbf{a} := ((c_0, d_0), ..., (c_n, d_n))$, the system (3.2) is equivalent to the following interpolation problem for the polynomial curve $\mathbf{P}_{\mathbf{a}} = (P_{\mathbf{c}}, P_{\mathbf{d}})$:

$$\mathbf{P}_{\mathbf{a}}(t_j) = (u_j, v_j), \qquad j=1, ..., 2n. \tag{3.3}$$

Since the vector $\mathbf{a}$ and the values t_j $(j=2, ..., 2n-1)$ are unknown, there are points (u_j, v_j), $j=1, ..., 2n$, such that the Eqs. (3.3) have no solution. Applying the implicit function theorem to F, we show that the system (3.3) is solvable in a neighborhood of a particular solution $(\mathbf{x}^0; (u_0^0, v_0^0), ..., (u_n^0, v_n^0))$. To this end, we now compute the determinant of the $4n \times 4n$ Jacobian matrix dF/dz; i.e., the ith row of dF/dz consists of the partial derivatives of the ith coordinate function of F.

3.1. THEOREM.

$$\left|\det\left(\frac{dF}{dz}\right)(\mathbf{x})\right| = \left|\operatorname{res}(P'_{\mathbf{c}}, P'_{\mathbf{d}}) \cdot \prod_{\substack{i,j=1\\ i<j}}^{2n} (t_j - t_i)\right|.$$

Primes indicate differentiation with respect to t, and $\operatorname{res}(P'_{\mathbf{c}}, P'_{\mathbf{d}})$ denotes the resultant of $P'_{\mathbf{c}}(t) = \sum_{i=0}^{n-1} (i+1)\, c_{i+1} t^i$ and $P'_{\mathbf{d}}(t)$ (see [16, p. 24]).

Proof.

$$\det\left(\frac{dF}{dz}\right)(\mathbf{x})$$

$$= \left|\begin{array}{ccccc|cccc|ccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0\cdots & 0 \\
1 & t_2 & t_2^2 & \cdots & t_2^n & 0 & & & 0 & P'_{\mathbf{c}}(t_2) & 0\cdots & 0 \\
\vdots & & & & \vdots & \vdots & & & \vdots & 0 & & \vdots \\
1 & t_{2n-1} & \cdots & \cdots & t_{2n-1}^n & 0 & & & 0 & \vdots & & P'_{\mathbf{c}}(t_{2n-1}) \\
1 & 1 & \cdots & \cdots & 1 & 0 & & & 0 & 0\cdots 0 & & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & & & & 0 & 1 & t_2 & \cdots & t_2^n & P'_{\mathbf{d}}(t_2) & 0\cdots & 0 \\
\vdots & & & & \vdots & \vdots & \vdots & & \vdots & 0 & & \vdots \\
0 & & & & 0 & 1 & t_{2n-1} & \cdots & t_{21n-1}^n & \vdots & & P'_{\mathbf{d}}(t_{2n-1}) \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0
\end{array}\right|$$

First, we expand this determinant by the first and the $(2n+1)$th row. Then, for $i = n-1, ..., 1$, we subtract the ith column from the $(i+1)$th column, and for $j = 2n-1, ..., n+1$ the jth column from the $(j+1)$th column. This yields the new determinant:

$$\pm\left|\begin{array}{ccc|ccc|ccc}
t_2 & t_2(t_2-1)\cdots & t_2^{n-1}(t_2-1) & 0 & \cdots & 0 & P'_{\mathbf{c}}(t_2) & 0\cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
t_{2n-1} & \cdots & t_{2n-1}^{n-1}(t_{2n-1}-1) & 0 & \cdots & 0 & 0\cdots & 0 & P'_{\mathbf{c}}(t_{2n-1}) \\
1 & 0\cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & t_2 & \cdots & t_2^{n-1}(t_2-1) & P'_{\mathbf{d}}(t_2) & 0\cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & t_{2n-1} & \cdots & t_{2n-1}^{n-1}(t_{2n-1}-1) & 0\cdots & 0 & P'_{\mathbf{d}}(t_{2n-1}) \\
0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0
\end{array}\right|$$

We now expand this determinant by the $(2n-1)$th and by the $(4n-2)$th row, then we multiply the ith row by $-P'_{\mathrm{d}}(t_{i+1})$, the $(i+2n-2)$th row by $P'_{\mathrm{c}}(t_{i+1})$, and add the $(i+2n-2)$th row to the ith row $(i=1, ..., 2n-2)$. Defining $\gamma_i := P'_{\mathrm{c}}(t_i) \cdot P'_{\mathrm{d}}(t_i)$, $\alpha := \prod_{i=2}^{2n-1} \gamma_i$, and $\beta := \prod_{i=2}^{2n-1} t_i \cdot (t_i - 1)$, we have obtained

$$\det\left(\frac{dF}{dz}\right)(\mathbf{x}) \cdot \alpha$$

$$= \beta \cdot \pm \begin{vmatrix} -P'_{\mathrm{d}}(t_2)\, t_2^0 & \cdots & -P'_{\mathrm{d}}(t_2)\, t_2^{n-2} & P'_{\mathrm{c}}(t_2)\, t_2^0 & \cdots & P'_{\mathrm{c}}(t_2)\, t_2^{n-2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ -P'_{\mathrm{d}}(t_{2n-1}) & \cdots & -P'_{\mathrm{d}}(t_{2n-1})\, t_{2n-1}^{n-2} & P'_{\mathrm{c}}(t_{2n-1}) & \cdots & P'_{\mathrm{c}}(t_{2n-1})\, t_{2n-1}^{n-2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & P'_{\mathrm{c}}(t_2)\, t_2^0 & \cdots & P'_{\mathrm{c}}(t_2)\, t_2^{n-2} & \gamma_2 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & P'_{\mathrm{c}}(t_{2n-1}) & \cdots & P'_{\mathrm{c}}(t_{2n-1})\, t_{2n-1}^{n-2} & 0 & \cdots & \gamma_{2n-1} \end{vmatrix}$$

$$= \alpha \cdot \beta \cdot \pm \begin{vmatrix} P'_{\mathrm{d}}(t_2)\, t_2^0 & \cdots & P'_{\mathrm{d}}(t_2)\, t_2^{n-2} & P'_{\mathrm{c}}(t_2) & \cdots & P'_{\mathrm{c}}(t_2)\, t_2^{n-2} \\ \vdots & & \vdots & \vdots & & \vdots \\ P'_{\mathrm{d}}(t_{2n-1}) & \cdots & P'_{\mathrm{d}}(t_{2n-1})\, t_{2n-1}^{n-2} & P'_{\mathrm{c}}(t_{2n-1}) & \cdots & P'_{\mathrm{c}}(t_{2n-1})\, t_{2n-1}^{n-2} \end{vmatrix}$$

$$= \alpha \cdot \beta \cdot \pm \left| \begin{bmatrix} d_1 & 2d_2 & \cdots & nd_n & 0 & \cdots & 0 \\ 0 & d_1 & 2d_2 & \cdots & nd_n & \cdots & 0 \\ & & \cdots & & & & \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & d_1 & 2d_2 & \cdots & nd_n \\ c_1 & 2c_2 & \cdots & nc_n & 0 & \cdots & 0 \\ 0 & c_1 & 2c_2 & \cdots & nc_n & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & c_1 & 2c_2 & \cdots & nc_n \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ t_2 & & t_{2n-1} \\ \vdots & & \vdots \\ t_2^{2n-3} & \cdots & t_{2n-1}^{2n-3} \end{bmatrix} \right|$$

The first determinant is the resultant of $P'_{\mathrm{c}}(t)$ and $P'_{\mathrm{d}}(t)$, the second determinant is the Vandermonde determinant, and the factor α may be cancelled, completing the proof. ▮

If two or more values t_i coincide, then $\det(dF/dz)(\mathbf{x}) = 0$. Using the following divided differences, we can transform the mapping F into an "equivalent" mapping G with nonzero Jacobian.

3.2. DEFINITION. Let $\mathbf{q}=(q_1, q_2): I \to \mathbb{R}^2$ be a regular curve. Given $r \in \mathbb{N}^+$ and $r+1$ points $t_i \in \mathbb{R}$ $(i=0, \ldots, r)$ such that $q_1(t_j) \neq q_1(t_k)$ for all $j \neq k$, we define, by recursion, for $k=0, \ldots, r-1$,

$$\Delta_0(\mathbf{q})[t_i] := q_2(t_i), \qquad i=0, \ldots, r,$$

and

$$\begin{aligned}\Delta_{k+1}(\mathbf{q})&[t_i, \ldots, t_{i+k+1}]\\ &:= (\Delta_k(\mathbf{q})[t_i, \ldots, t_{i+k}] - \Delta_k(\mathbf{q})[t_{i+1}, \ldots, t_{i+k+1}])/(q_1(t_i) - q_1(t_{i+k+1})),\\ &\quad i=0, \ldots, r-k-1.\end{aligned}$$

$\Delta_r(\mathbf{q})$ is considered to be a function from the set $D := \{(t_0, \ldots, t_r) \in \mathbb{R}^{r+1} \mid \Delta_r(\mathbf{q})[t_0, \ldots, t_r]$ is defined$\}$ into $\mathbb{R}$. If q_1 is strictly monotone on an open or closed interval $J \subset I$ such that $t_i \in J$ for $i=0, \ldots, r$, then $\Delta_r(\mathbf{q})[t_0, \ldots, t_r]$ is equal to the rth divided difference of the real function $q_2 \circ (q_1 \mid J)^{-1}$ for the arguments $q_1(t_0), \ldots, q_1(t_r)$. Hence we obtain from [13, Chap. I, 1.8]:

3.3. PROPOSITION. *Let $\mathbf{q}=(q_1, q_2)$: $I \to \mathbb{R}^2$ be a C^r curve. Given an open or closed subinterval J of I such that $q_1'(t) \neq 0$ for every $t \in J$, there is a unique continuous extension of $\Delta_r(\mathbf{q})$ onto $D \cup J^{r+1}$, and $\Delta_r(\mathbf{q})[t, \ldots, t] = (d^r/ds^r)(q_2 \circ q_1^{-1})(q_1(t))/r!$ $(t \in J)$.*

Contact of order r can be characterized by divided differences as follows:

3.4. PROPOSITION. *Let $\mathbf{q}=(q_1, q_2)$: $I \to \mathbb{R}^2$ and $\bar{\mathbf{q}}=(\bar{q}_1, \bar{q}_2)$: $I \to \mathbb{R}^2$ be two regular C^r curves, and let $s \in \{0, 1\}$ such that $\mathbf{q}(s)=\bar{\mathbf{q}}(s)$ and $q_1'(s) \cdot \bar{q}_1'(s) > 0$. Then $\mathbf{q}$ and $\bar{\mathbf{q}}$ have contact of order $r \geqslant 1$ at s if and only if $\Delta_k(\bar{\mathbf{q}})[s_0, \ldots, s_k] = \Delta_k(\mathbf{q})[s_0, \ldots, s_k]$ for $k=0, \ldots, r$, where $s_i := s$ for $k = 0, \ldots, r$.*

Proof. Let $q_1'(s) > 0$ (in case $q_1'(s) < 0$ use the transformation of coordinates $(x_1, x_2) \in \mathbb{R}^2 \mapsto (-x_1, x_2)$). Then there exists a neighborhood U of s in I such that $q_1'(t) > 0$ and $\bar{q}_1'(t) > 0$ for every $t \in U$. Since contact of order r is a local property, we may consider the restricted curves $\mathbf{q} \mid_U$ and $\bar{\mathbf{q}} \mid_U$. Using the orientation preserving local reparametrizations $\varphi := (q_1 \mid_U)^{-1}$ and $\bar{\varphi} := (\bar{q}_1 \mid_U)^{-1}$, we get equivalent parametrizations $t \mapsto (t, q_2(\varphi(t)))$ and $t' \mapsto (t', \bar{q}_2(\bar{\varphi}(t')))$ of $\mathbf{q} \mid_U$ and $\bar{\mathbf{q}} \mid_U$, respectively, in a neightborhood V of $q_1(s)=\bar{q}_1(s)$. By [3, p. 88, A3.3, Ex. 1], contact of order r of graphs of functions $f := q_2 \circ \varphi$ and $g := \bar{q}_2 \circ \bar{\varphi}$ at $s' := q_1(s)$ is equivalent to $f^{(k)}(s') = g^{(k)}(s')$ for $k=0, \ldots, r$. Proposition 3.3 now implies the desired result. ∎

4. Proofs of Theorems 2.6 and 2.7

For any $k \in \{0, ..., n-2\}$, it is easier to establish the local solvency of $E|_{(B_k \times I)}$ than to establish the local Property Z at a point $\mathbf{a} \in B_k$. Both proofs can be based on the system of Eqs. (3.2) and the implicit function theorem.

Proof of 2.6. (1) Let $k := 0$ and $U \neq \emptyset$ be any open subset of B. Moreover, let there be given $\mathbf{a}_0 \in B_0 \cap U$, $\varepsilon > 0$, and $2n$ values $s_i \in I$ with $s_1 := 0 < s_2 < \cdots < s_{2n} := 1$. By Definition 2.1, we have to find $\delta > 0$ such that the $2n-2$ equations $E(\mathbf{a}, s_i) = y_i$, $i = 2, ..., 2n-1$, have a solution $\mathbf{a} \in B_0 \cap U$ with $|E(\mathbf{a}, \cdot) - E(\mathbf{a}_0, \cdot)|_\infty < \varepsilon$ if $y_i \in \mathbb{R}$ and $|y_i - E(\mathbf{a}_0, s_i)| < \delta$ for $i = 2, ..., 2n-1$. By definition of E, we have to solve the $2n$ equations (note that E and ϕ are not explicitly known):

$$\mathbf{P}_{\mathbf{a}}(t_i) := \mathbf{p}(s_i) + y_i \cdot \mathbf{n}(s_i) =: (u_i, v_i), \qquad i = 1, ..., 2n, \qquad (*)$$

where $y_1 := y_{2n} := 0$, $t_1 := \phi(\mathbf{a}, s_1) = 0$, and $t_{2n} := \phi(\mathbf{a}, s_{2n}) = 1$, i.e., the points (u_i, v_i), $i = 1, ..., 2n$, $t_1 = 0$, and $t_{2n} = 1$ are given and $\mathbf{a} \in B_0 \cap U$ and $t_i \in I$, $i = 2, ..., 2n-1$, are unknown. Thus we have obtained Eqs. (3.3) or the equivalent system (3.2), where $((c_0, d_0), ..., (c_n, d_n)) := \mathbf{a}$. Furthermore, if $y_i := E(\mathbf{a}_0, s_i)$, $i = 1, ..., 2n$, then $\mathbf{a}_0$ and the values $\phi(\mathbf{a}_0, s_i)$, $i = 2, ..., 2n-1$, solve (3.2) and (*). This solution is denoted by $\mathbf{x}_0$. Moreover, note that if the points $\mathbf{a} \in \mathbb{R}^{2n+2}$ and $t_i \in I$, $i = 2, ..., 2n-1$, solve (*), and if the values $|\mathbf{a} - \mathbf{a}_0|_2$, $|\phi(\mathbf{a}_0, s_i) - t_i|$, and $|y_i - E(\mathbf{a}_0, s_i)|$, $i = 2, ..., 2n-1$, are sufficiently small, then we may conclude that $\mathbf{a} \in U \subset B$, $\mathbf{a} \in B_0$ (because $\mathbf{P}_{\mathbf{a}}(x) = \mathbf{p}(x)$ for $x = 0, 1$), $t_i = \phi(\mathbf{a}, s_i)$, and $y_i = E(\mathbf{a}, s_i)$ for $i = 2, ..., 2n-1$, because ϕ and E are continuous and, by Definition 1.2(3), each normal $N(s_i)$ meets $\mathbf{P}_{\mathbf{a}}$ in a unique point in a neighborhood of $\mathbf{P}_{\mathbf{a}_0}(\phi(\mathbf{a}_0, s_i))$.

Since $s_i \neq s_j$ for $i \neq j$ and $\mathbf{a}_0 \in B_0 \subset A$ (see (1.1)), it follows from 3.1 that the Jacobian determinant of F does not vanish at $\mathbf{x}_0$, and hence the assertion follows from the implicit (or inverse) function theorem.

(2) Let $k > 0$, $\mathbf{a}_0 \in B_k$, $\varepsilon > 0$, and let there be given $2n$ values $s_i \in I$ such that $s_1 := \cdots := s_{k+1} := 0 < s_{k+2} < \cdots < s_{2n-k} := \cdots := s_{2n} := 1$. Following along the same lines as in part (1) of this proof, we obtain the system of equations (*), where $y_1 := \cdots := y_{k+1} := y_{2n-k} := \cdots := y_{2n} := 0$, $t_1 := \phi(\mathbf{a}, s_1) = 0$, and $t_{2n} := \phi(\mathbf{a}, s_{2n}) = 1$. Using the equivalent system of equations (3.2) instead of (*), we have the same solution $\mathbf{x}_0$ as in part (1), but, by 3.1 and assumption,$\det(dF/dz)(\mathbf{x}_0) = 0$ and (3.2) does not imply $\mathbf{a} \in B_k$. Therefore, for $r = 1, ..., k$, we replace the equations $F_{2,r+1}(\mathbf{x})(= P_{\mathbf{d}}(t_{r+1})) =$

v_{r+1} and $F_{2,2n-r}(\mathbf{x})(=P_{\mathbf{d}}(t_{2n-r}))=v_{2n-r}$ with $v_{r+1}:=p_2(0)$ and $v_{2n-r}:=p_2(1)$ by the equations (cf. Definition 3.2):

$$\Delta_r(P_{\mathbf{c}}, P_{\mathbf{d}})[t_1, ..., t_{r+1}] = \Delta_r(\mathbf{p})[0, ..., 0]$$

and (**)

$$\Delta_r(P_{\mathbf{c}}, P_{\mathbf{d}})[t_{2n-r}, ..., t_{2n}] = \Delta_r(\mathbf{p})[1, ..., 1],$$

respectively (note $\mathbf{p}=(p_1, p_2)$). Replacing the coordinate functions $F_{2,r+1}$ and $F_{2,2n-r}$ of F by the functions $\Delta_r(P_{\mathbf{c}}, P_{\mathbf{d}})[t_1, ..., t_{r+1}]$ and $\Delta_r(P_{\mathbf{c}}, P_{\mathbf{d}})[t_{2n-r}, ..., t_{2n}]$, respectively, $r=1, ..., k$, we obtain a new mapping G_k. By 3.3, G_k is of class C^1 in a neighborhood of $\mathbf{x}_0$, if $p_1'(0)\neq 0$ and $p_1'(1)\neq 0$ (otherwise choose another Cartesian coordinate system in $\mathbb{R}^2$). Furthermore, equations $P_{\mathbf{c}}(t_i)=p_1(0)$, $i=1, ..., k+1$, and $P_{\mathbf{c}}(t_j)=p_1(1)$, $j=2n-k, ..., 2n$, with $t_1=0$, $t_{2n}=1$ imply $t_1=\cdots=t_{k+1}=0$ and $t_{2n-k}=\cdots=t_{2n}=1$ if $|\mathbf{x}-\mathbf{x}_0|_2$ is sufficiently small. Therefore, we may conclude from Eqs. (**) and Proposition 3.4 that $\mathbf{a}\in B_k$ if $|\mathbf{x}-\mathbf{x}_0|_2$ is sufficiently small (note B is open). By the first part of this proof, it remains to show that the Jacobian of G_k does not vanish at $\mathbf{x}_0$. This follows from the definition of divided differences and Lemma 4.1 below, because the factors t_i-t_j can be cancelled in $\det(dF/dz)$ for $1\leqslant i<j\leqslant k+1$ and $2n-k\leqslant i<j<2n$. For instance, if $k=1$, then $\det(dG/dz)(\mathbf{x}_0)=\det(dF/dz)(\mathbf{x}_0)/(P_{\mathbf{c}}'(0)\cdot P_{\mathbf{c}}'(0)\cdot(t_2-t_1)\cdot(t_{2n}-t_{2n-1}))$. The details are left to the reader, completing the proof. ∎

4.1. Lemma. *Let $s\in\mathbb{N}$, $i, j, i', j'\in\{1, ..., s\}$ with $i\neq j$ and $i'\neq j'$, and let $f_1, ..., f_s$, $g_1, ..., g_s$ be continuously differentiable functions from an open subset U of $\mathbb{R}^{2s}$ into $\mathbb{R}$. Put $H:=(f_1, ..., f_s, g_1, ..., g_s)$ and $H':=(f_1, ..., f_{j-1}, (f_j-f_i)/(g_{j'}-g_{i'}), f_{j+1}, ..., f_s, g_1, ..., g_s)$, where H' is defined on the set $D:=\{z\in U \mid g_{i'}(z)\neq g_{j'}(z)\}$. Then $\det(dH'/dz)(a)=\det(dH/dz)(a)/(g_{j'}(a)-g_{i'}(a))$ for every $a\in D$.*

Proof. For each $k\in\{1, ..., 2s\}-\{j\}$ and $a\in D$ the kth row of $A:=(dH/dz)(a)$ is equal to the kth row of $B:=(dH'/dz)(a)$. Furthermore, the kth component of the jth row of B is equal to

$$\frac{(\partial f_j/\partial z_k-\partial f_i/\partial z_k)}{(g_{j'}-g_{i'})}(a)-\frac{(\partial g_{j'}/\partial z_k-\partial g_{i'}/\partial z_k)\cdot(f_j-f_i)}{(g_{j'}-g_{i'})^2}(a).$$

Therefore, the jth row of B is a linear combination of the vector $v:=(g_{j'}-g_{i'})^{-1}(a)\cdot((\partial f_j/\partial z_1), ..., (\partial f_j/\partial z_{2s}))(a)$ and of the ith, $(s+i')$th, and $(s+j')$th row of B. From this we derive that the jth row of $\det(B)$ may be replaced by v, completing the proof. ∎

Indirect proof of 2.7. (1) Let $k := 0$ and $\mathbf{a}_0 \in B_0$. We assume that there is no neighborhood $W \subset \mathbb{R}^{2n+2}$ of $\mathbf{a}_0$ such that $E|_{(U \times I)}$ with $U := W \cap B_k$ has Property Z of degree $2 \cdot (n-1)$ at each $\mathbf{b} \in U$ w.r. to $]0, 1[$. Then there are two sequences $(\mathbf{a}_m)_{m \in \mathbb{N}}$ and $(\mathbf{b}_m)_{m \in \mathbb{N}}$ in B_k converging to the point $\mathbf{a}_0$ such that $\mathbf{a}_m \neq \mathbf{b}_m$ and $E(\mathbf{a}_m, \cdot) - E(\mathbf{b}_m, \cdot)$ has at least $2(n-1)$ zeros. Hence, for each $m \in \mathbb{N}$, there exist $2(n-1)$ values $s_i^{(m)}$ $(i = 2, ..., 2n-1)$ with $0 < s_2^{(m)} < \cdots < s_{2n-1}^{(m)} < 1$ such that for $i = 2, ..., 2n-1$:

$$E(\mathbf{a}_m, s_i^{(m)}) = E(\mathbf{b}_m, s_i^{(m)}) \tag{*}$$

and hence

$$\mathbf{P}_{\mathbf{a}_m}(\phi(\mathbf{a}_m, s_i^{(m)})) = \mathbf{P}_{\mathbf{b}_m}(\phi(\mathbf{b}_m, s_i^{(m)})).$$

Since $I^{2(n-1)}$ is compact, there is a subsequence of $(s_2^{(m)}, ..., s_{2n-1}^{(m)})_{m \in \mathbb{N}}$ converging to a vector $(s_2, ..., s_{2n-1})$ with $0 \leqslant s_2 \leqslant \cdots \leqslant s_{2n-1} \leqslant 1$ (see [6, p. 217, 6.1 and p. 229, 3.2]).

Now let $\mathbf{a}_0 := ((c_0^0, d_0^0), ..., (c_n^0, d_n^0))$, $\mathbf{x}_0 = (c_0^0, ..., c_n^0, d_0^0, ..., d_n^0, t_2^0, ..., t_{2n-1}^0)$, where $t_i^0 := \phi(\mathbf{a}_0, s_i)$, $i = 1, ..., 2n$, with $s_1 := 0$ and $s_{2n} := 1$. By (*), there is no neighborhood V of $\mathbf{x}_0$ such that $F|_V$ (see (3.1)) is injective (note that (3.2) is equivalent to (3.3)). If $0 < s_2 < \cdots < s_{2n-1} < 1$, then, by 3.1, $\det(dF/dz)(\mathbf{x}_0) \neq 0$ and so, by the implicit (or inverse) function theorem, F is one-to-one in some neighborhood of $\mathbf{x}_0$. This is a contradiction.

If two or more values s_i coincide, then, using divided differences, the mapping F can be transformed into a C^1 mapping H such that H has a nonzero Jacobian at $\mathbf{x}_0$; but, because of (*), H is not one-to-one in any neighborhood of $\mathbf{x}_0$, a contradiction. We treat a special case and the general case is left to the reader.

We assume $n > 3$ and $0 = s_1 = s_2 = s_3 < s_4 < \cdots < s_{2n-2} = s_{2n-1} < 1$. Choosing another Cartesian coordinate system of $\mathbb{R}^2$ if necessary, we may assume $P'_{\mathbf{c}}(s_1) \neq 0$ and $P'_{\mathbf{c}}(s_{2n-2}) \neq 0$. We now define:

$$H(\mathbf{x}) := (F_{1,1}, ..., F_{1,2n}, F_{2,1}, \Delta_1(P_{\mathbf{c}}, P_{\mathbf{d}})[t_1, t_2], \Delta_2(P_{\mathbf{c}}, P_{\mathbf{d}})[t_1, t_2, t_3],$$

$$F_{2,4}, ..., F_{2,2n-3}, \Delta_1(P_{\mathbf{c}}, P_{\mathbf{d}})[t_{2n-2}, t_{2n-1}], F_{2,2n-1}, F_{2,2n}).$$

As in the proof of 2.6, part (2), it follows from the definition of divided differences, Theorem 3.1, and Lemma 4.1 that the mapping H has the desired properties. This completes the proof for $k = 0$.

(2) Let $k > 0$. Following along the same lines as for $k = 0$, we may assume that there are sequences $(\mathbf{a}_m)_{m \in \mathbb{N}}$ and $(\mathbf{b}_m)_{m \in \mathbb{N}}$ in B_k converging to

$\mathbf{a}_0 \in B_k$, $2n-2k-2$ sequences $(s_i^{(m)})_{m \in \mathbb{N}}$, and values s_i, $i=k+2, ..., 2n-k-1$, such that

$$\mathbf{P}_{\mathbf{a}_m}(\phi(\mathbf{a}_m, s_i^{(m)})) = \mathbf{P}_{\mathbf{b}_m}(\phi(\mathbf{b}_m, s_i^{(m)})), \qquad i=k+2, ..., 2n-k-1 \text{ and } m \in \mathbb{N}, \tag{**}$$

$0 < s_i^{(m)} < s_{i+1}^{(m)} < 1$, and a subsequence of $(s_i^{(m)})_{m \in \mathbb{N}}$ converges to s_i. Since $\mathbf{a}_m$, $\mathbf{b}_m \in B_k$, we obtain from 3.4

$$\Delta_r(\mathbf{P}_{\mathbf{a}_m})[u, ..., u] = \Delta_r(\mathbf{P}_{\mathbf{b}_m})[u, ..., u], \qquad r=0, ..., k \text{ and } u \in \{0, 1\}, \tag{***}$$

where we have supposed that $p_1'(0) \neq 0$ and $p_1'(1) \neq 0$. If the mapping G_k is defined as in part (2) of the proof of 2.6, then we conclude from (**) and (***) that there is no neighborhood V of the above solution $\mathbf{x}_0$ with $s_1 := \cdots := s_{k+1} := 0$ and $s_{2n-k} := \cdots := s_{2n} := 1$ such that $G_k \mid_V$ is one-to-one. If $0 < s_{k+1} < \cdots < s_{2n-k-1} < 1$, then $\det(dG_k/dz)(\mathbf{x}_0) \neq 0$ and we have a contradiction. Otherwise G_k first must be transformed by divided differences as above. In particular, if $0 = s_{k+1} = s_{k+2}$ or $1 = s_{2n-k} = s_{2n-k-1}$, then one must use confluent divided differences (see [13, p. 13]). The details are left to the reader. ∎

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